

# Quantum particle on hyperboloid

Włodzimierz Piechocki

*Soltan Institute for Nuclear Studies, Hoża 69,  
00-681 Warszawa, Poland; e-mail: piech@fuw.edu.pl*

## Abstract

We present quantization of particle dynamics on one-sheet hyperboloid embedded in three dimensional Minkowski space. Taking account of all global symmetries of the system enables unique quantization. Making use of topology of canonical variables not only simplifies calculations but also gives proper framework for analysis.

PACS numbers: 04.60.Ds, 02.20.Qs, 11.30.Fs

arXiv:gr-qc/0308030v1 11 Aug 2003

## I. INTRODUCTION

It is known that canonical quantization of a system with non-trivial topology of its phase space (i.e. different from  $\mathbb{R}^{2n}$ ) is a nonunique procedure (see, e.g. [1, 2]). It has been found that ambiguities of quantization may be reduced by taking into account global properties of a system [3, 4].

In [4] we have quantized the dynamics of a relativistic test particle on a one-sheet hyperboloid embedded in three dimensional Minkowski space, i.e. on the two dimensional de Sitter space with the topology  $\mathbb{R} \times \mathbb{S}$ . The resulting quantum theory depends on a real parameter  $\theta$ . To simplify the discussion we fixed the parameter to its values  $\theta = 0$ , which corresponds to the choice  $SO_0(1, 2)$  as the symmetry group of the system. In the present work we discuss the  $\theta$ -dependence of the results. We show that making use of time-reversal invariance of the system fixes the parameter to its two values:  $\theta = 0$  and  $\theta = \pi$ .

In our quantization procedure we make use of the idea introduced in [5] of taking  $U(\beta) := \exp(i\beta)$ ,  $0 \leq \beta < 2\pi$ , to represent the variable with topology  $\mathbb{S}$ , instead of the common choice  $\tilde{U}(\beta) := \beta$ . This idea has proved to be fruitful in the quantization of particle dynamics on a circle [5] and on a sphere [6].

## II. OBSERVABLES

The system of a relativistic test particle on a one-sheet hyperboloid

$$(y^2)^2 + (y^1)^2 - (y^0)^2 = r_0^2, \quad (1)$$

where  $y^a$  ( $a = 0, 1, 2$ ) denote coordinates of 3-dimensional Minkowski space and  $r_0$  is the parameter specifying the hyperboloid, has been found to be integrable [7].

The space of timelike geodesics of a particle may be determined by the solutions to the algebraic equations [7]

$$J_a y^a = 0, \quad J_2 y^1 - J_1 y^2 = r_0 p, \quad (2)$$

where  $p$  denotes one of the two canonical momenta of a particle. The dynamical integrals  $J_a$  ( $a = 0, 1, 2$ ), owing to the constrained dynamics of the system, satisfy the equation [7]

$$J_2^2 + J_1^2 - J_0^2 = \kappa^2, \quad \kappa := m_0 r_0, \quad (3)$$

where  $m_0$  denotes particle's mass.  $J_a$  ( $a = 0, 1, 2$ ) are generators of the proper orthochronous Lorentz group  $SO_0(1, 2)$ ;  $J_0$  corresponds to the invariance of (1) with respect to rotations, whereas  $J_1$  and  $J_2$  describe two boosts. The generators of  $SO_0(1, 2)$  group are basic observables of the system [4].

The phase space  $\Gamma$  of the system is defined to be the space of all timelike geodesics of a particle available for dynamics. Each point  $(J_0, J_1, J_2)$  of the hyperboloid (3) specifies a geodesic with coordinates satisfying (1) and (2). Thus, the hyperboloid (3) plays the role of  $\Gamma$ . The system has two degrees of freedom and  $\Gamma$  (in what follows being identified with (3)) is globally homeomorphic to the canonical phase space  $X$  defined as

$$X := \{(J, \beta) \mid J \in \mathbb{R}, \quad 0 \leq \beta < 2\pi\}, \quad (4)$$

where the homeomorphism may be defined [4] by

$$J_0 = J, \quad J_1 = J \cos \beta - \kappa \sin \beta, \quad J_2 = J \sin \beta + \kappa \cos \beta. \quad (5)$$

Introducing the Poisson bracket on  $X$  by

$$\{\cdot, \cdot\} := \frac{\partial \cdot}{\partial J} \frac{\partial \cdot}{\partial \beta} - \frac{\partial \cdot}{\partial \beta} \frac{\partial \cdot}{\partial J}, \quad (6)$$

we obtain

$$\{J_0, J_1\} = -J_2, \quad \{J_0, J_2\} = J_1, \quad \{J_1, J_2\} = J_0, \quad (7)$$

which means that the observables  $J_a$  ( $a = 0, 1, 2$ ) satisfy the algebra  $so(1, 2)$ . The algebra (7) describes local symmetry of the physical phase space (3). Since the topology of the hyperboloid (3) is  $\mathbb{R} \times \mathbb{S}$ , the global symmetry of  $\Gamma$  may be taken to be any group with  $so(1, 2)$  as its Lie algebra, i.e.  $SO_0(1, 2)$  or any covering of it.

Untill now we have considered only *continuous* transformations, but our system may be also invariant under *discrete* transformations. Since the system of a particle on hyperboloid is a non-dissipative one, it must be invariant with respect to time-reversal transformations  $T$ . We postpone further discussion of  $T$ -invariance to the section dealing with quantization.

### III. REDEFINITIONS

It is advantageous, for further considerations, to redefine the basic observables (5) and the canonical variables (4) as follows

$$J_0 := J, \quad J_+ := J_1 + iJ_2 = (J + i\kappa)U, \quad J_- := J_1 - iJ_2 = (J - i\kappa)U, \quad (8)$$

where  $U := \exp(i\beta)$ ,  $0 \leq \beta < 2\pi$ .

The canonical phase space  $X$  is now represented by

$$X = \{(J, U) \mid J \in \mathbb{R}, U \in \mathbb{S}\}. \quad (9)$$

It has been shown in [5] that making use of  $U$ , instead of  $\beta$ , to represent the variable with  $\mathbb{S}$  topology is mathematically better justified.

We also redefine the algebra multiplication replacing (6) by

$$\ll \cdot, \cdot \gg := \left( \frac{\partial \cdot}{\partial J} \frac{\partial \cdot}{\partial U} - \frac{\partial \cdot}{\partial U} \frac{\partial \cdot}{\partial J} \right) U = \{\cdot, \cdot\} U. \quad (10)$$

One can check that owing to the Poisson bracket properties the bracket (10) satisfies the three axioms: linearity, antisymmetry and the Jacobi identity. Thus (10) defines the Lie multiplication.

One can easily verify that the algebra (7) in new disguise reads

$$\ll J_0, J_+ \gg = J_+, \quad \ll J_0, J_- \gg = -J_-, \quad \ll J_-, J_+ \gg = 2J_0. \quad (11)$$

In particular, the new canonical variables satisfy the algebra

$$\ll J, U \gg = U \quad (12)$$

to be compared with the algebra

$$\{J, \beta\} = 1 \quad (13)$$

based on (4) and (6).

## IV. QUANTIZATION

By quantization of particle dynamics we mean finding an (essentially) self-adjoint representation of the algebra of observables integrable to an irreducible unitary representation of the symmetry group of the system. The representation space plays the role of the quantum states space.

To take into account the time-reversal invariance of the system we require the representation algebra to be invariant under time-reversal operator representing time-reversal transformations.

### A. Representation of canonical variables algebra

To quantize the algebra (11), we first quantize the canonical variables algebra (12). Following the idea presented in [5] we make use of the mapping

$$J \rightarrow \hat{J}\psi(\beta) := -i\frac{d}{d\beta}\psi(\beta), \quad U = e^{i\beta} \rightarrow \hat{U}\psi(\beta) := e^{i\hat{\beta}}\psi(\beta) := e^{i\beta}\psi(\beta), \quad (14)$$

where  $\psi \in L^2(\mathbb{S})$ .

It is easy to check (see, App. A of [4]) that the operator  $\hat{J}$  is essentially self-adjoint on the dense subspace  $\Omega_\theta$ ,  $0 \leq \theta < 2\pi$ , of  $L^2(\mathbb{S})$  defined to be

$$\Omega_\theta := \{\psi \in L^2(\mathbb{S}) \mid \psi \in C^\infty[0, 2\pi], \psi^{(n)}(2\pi) = e^{i\theta}\psi^{(n)}(0), n = 0, 1, 2, \dots\}. \quad (15)$$

It follows easily that the spectrum of  $\hat{J}$  reads

$$sp\hat{J}(\theta) = \{j := m + \frac{\theta}{2\pi} \mid m \in \mathbb{Z}, 0 \leq \theta < 2\pi\}, \quad (16)$$

and  $\Omega_\theta$  is spanned by

$$f_{m,\theta}(\beta) = \frac{1}{2\pi} e^{i\beta(m + \frac{\theta}{2\pi})}. \quad (17)$$

We see at once that

$$[\hat{J}, \hat{U}]\psi := \hat{J}\hat{U}\psi - \hat{U}\hat{J}\psi = \widehat{\llbracket J, U \rrbracket}\psi = \hat{U}\psi, \quad \psi \in \Omega_\theta, \quad (18)$$

since the unitary operator  $\hat{U}$  is well defined on the entire  $L^2(\mathbb{S})$ . Therefore, the mapping (14) leads to the essential self-adjoint representation of (12) on  $\Omega_\theta$ . Since the parameter  $\theta$  is a real number, the algebra (12) has an infinite number of unitarily nonequivalent representations.

### B. Representation of observables

Applying (14) and the symmetrisation prescription to the products  $JU$  in (8) we obtain the following mapping

$$J_0 \rightarrow \hat{J}_0 := \hat{J}, \quad J_- \rightarrow \hat{J}_- := \hat{U}^{-1}(\hat{J} - 1/2 - i\kappa), \quad J_+ \rightarrow \hat{J}_+ := (\hat{J} - 1/2 + i\kappa)\hat{U}, \quad (19)$$

where  $\hat{U}^{-1} := \exp(-i\hat{\beta})$ .

One can easily verify that (19) is a homomorphism

$$[\hat{J}_0, \hat{J}_+] = \ll \widehat{J_0, J_+} \gg = \hat{J}_+, \quad [\hat{J}_0, \hat{J}_-] = \ll \widehat{J_0, J_-} \gg = -\hat{J}_-, \quad [\hat{J}_-, \hat{J}_+] = \ll \widehat{J_-, J_+} \gg = 2\hat{J}_0. \quad (20)$$

The operators (19) and the equations (20) are well defined on  $\Omega_\theta$ .

Making use of the method applied in App. A of [4] it is straightforward to prove that (19) defines the essentially self-adjoint representation of the algebra (11), if the common domain of the algebra (20) coincides with  $\Omega_\theta$ . The proof rests heavily on the fact that on  $\Omega_\theta$  the problem reduces to the problem of self-adjointness of the representation algebra (18).

### C. Time-reversal invariance

We impose the time-reversal invariance (see, e.g. [8]) upon the system by the requirement of time-reversal invariance of the algebra (20). In what follows we show that (20) is time-reversal invariant, if the algebra of canonical variables (18) has this property.

The observable  $J$  being associated with the invariance of (1) with respect to rotations may be given the interpretation of angular momentum of a test particle. Thus, the operator  $\hat{J}$  transforms as

$$\hat{T}\hat{J}\hat{T}^{-1} = -\hat{J}, \quad (21)$$

where  $\hat{T}$  is an anti-unitary operator of time inversion.

It is easy to check that the algebra (18) is time-reversal invariant [5], if the operator  $\hat{U}$  transforms as

$$\hat{T}\hat{U}\hat{T}^{-1} = \hat{U}^{-1}. \quad (22)$$

Owing to (21), (22) and (19) it is easy to check that

$$\hat{T}\hat{J}_+\hat{T}^{-1} = -\hat{U}^{-1}\hat{J}_+\hat{U}^{-1}, \quad \hat{T}\hat{J}_-\hat{T}^{-1} = -\hat{U}\hat{J}_-\hat{U}. \quad (23)$$

Making use of (23) one can easily show that the algebra (20) is time-reversal invariant.

Now, let us examine the consequences of the imposition of time-reversal invariance in the context of the representation space. It was shown in [5] that the eigenvalue equation

$$\hat{J}|j\rangle = j|j\rangle \quad (24)$$

combined with (18) leads to

$$\hat{U}|j\rangle = |j+1\rangle, \quad \hat{U}^{-1}|j\rangle = |j-1\rangle, \quad (25)$$

which means that the set of all eigenfunctions  $\{|j\rangle\}$  can be generated from the ‘vacuum’ state  $|j_0\rangle$ , where  $0 \leq j_0 \leq 1$ . It is a simple matter to show that the consistency of Eqs. (21), (22) and (25) yields [5]

$$\hat{T}|j\rangle = |-j\rangle. \quad (26)$$

Therefore, the spectrum of  $\hat{J}$  must be symmetric with respect to  $j$ , which means that  $j_0 = 0$  or  $j_0 = 1/2$ . Hence,  $j$  can be only integer or half-integer. Comparing this result with (16) and (17), we conclude that the range of the parameter  $\theta$  must be restricted to the two values:  $\theta = 0$  and  $\theta = \pi$ .

This way we have proved that the algebra (11) has two classes of representations. They are labeled by  $\theta = 0$  and  $\theta = \pi$ .

## D. Identification of representations

For clarity, we recall the existence of the following algebra isomorphisms [9]:  $so(1, 2) \sim sl(2, \mathbb{R}) \sim su(1, 1)$ , and group isomorphisms [11, 12]:  $SL(2, \mathbb{R}) \sim SU(1, 1)$  and  $SO_0(1, 2) \sim SL(2, \mathbb{R})/\mathbb{Z}_2 \sim \widetilde{SL(2, \mathbb{R})}/\mathbb{Z}$ .

To compare our representations with known representations of the Lie groups having mentioned above isomorphic Lie algebras, we consider the Casimir operator of the algebra (11). Its classical and quantum versions, respectively, read [4]

$$C := J_2^2 + J_1^2 - J_0^2 = J_+ J_- - J^2 = \kappa^2, \quad \hat{C} := \frac{1}{2}(\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+) - \hat{J}^2 = (\kappa^2 + 1/4)\mathbb{I}. \quad (27)$$

One can check at once that  $[\hat{C}, \hat{J}] = 0$ , which means that the representation space may be labelled by the three parameters:  $\theta$ ,  $\kappa$  and  $j$ .

Since  $1/4 < \kappa < \infty$  the class with  $\theta = 0$  and  $j$  integer, and the class with  $\theta = \pi$  and  $j$  half-integer belong to the principal series single valued and double valued, respectively, unitary representation of  $SU(1, 1)$  group [10, 11]. The single valued case coincides with the representation of  $SO_0(1, 2)$  group [10].

Therefore, the principal series representation of  $SL(2, \mathbb{R})$  group unifies the principal series representation of  $SO_0(1, 2)$  group and the representation of the time-reversal transformations.

The general case  $0 \leq \theta < 2\pi$  corresponds [12] to the principal series representation of the universal covering group  $\widetilde{SL(2, \mathbb{R})}$ .

## V. CONCLUSIONS

Quantization based on *local* properties of a classical system only, leads to nonuniqueness. We have seen that quantization of the local symmetry without the requirement of its integrability to the global symmetry yields infinitely many quantum systems corresponding to a single classical system. Such approach is also not suitable for quantization of systems with singular spacetimes [3, 4]. *Global* symmetries (continuous and discrete) turn out to be of primary importance. Making use of them we have managed to avoid problems in quantum theory connected with removable type singularities of spacetime [4] and have succeeded to get unique results. Obviously, we have not discussed here the commonly known ambiguity problem (see, e.g. [13]) connected with the imposition of quantum rules on constrained dynamics (quantize first then impose constraints or vice versa problem). But this problem is beyond the scope of the present paper.

We have found that the problem of self-adjointness of observables and time-reversal invariance of the system may be already implemented at the level of *canonical variables* algebra. This is possible owing to the choice of canonical variables in the form which fits the *topology* of the variables. The common choice of variables on  $\mathbb{S}$  in the form used for an interval  $[a, b] \subset \mathbb{R}$ , makes impossible the solution of the above problems at the canonical level. It is so because one cannot find a self-adjoint representation of the algebra (see, e.g. App. B of [14]). It may be solved, but at the level of the observables algebra (see, e.g. App. A of [4]). The choice of the canonical variables in the form compatible with their topologies seems to have basic significance. Finally, we make comment concerning our choice of the Lie multiplication (10). Comparing (18) with (12), and (20) with (11) we cannot see the factor

$(-i)$ , which usually occurs in homomorphisms from classical to quantum algebras. This is a nice feature of (10) and it is again connected with the topology consistent form of canonical variables.

## Acknowledgments

The author would like to thank Professor J. Rembieliński for very fruitful suggestions concerning the solution of the ambiguity problem.

- 
- [1] Isham C J 1984 Topological and global aspects of quantum theory *Relativity, Groups and Topology II* Eds B S DeWitt and R Stora (Amsterdam: North-Holland)
  - [2] Bojowald M and Strobl T 2000 *J. Math. Phys.* **41** 2537
  - [3] Piechocki W 2002 *Phys. Lett. B* **526** 127
  - [4] Piechocki W 2003 *Class. Quantum Grav.* **20** 2491
  - [5] Kowalski K, Rembieliński J and Papaloucas L C 1996 *J. Phys. A: Math. Gen.* **29** 4149
  - [6] Kowalski K and Rembieliński J 2000 *J. Phys. A: Math. Gen.* **33** 6035
  - [7] Jorjadze J and Piechocki W 1999 *Phys. Lett. B* **461** 183
  - [8] Tung W-K 1985 *Group Theory in Physics* (Singapore: World Scientific)
  - [9] Barut A O and Rączka R 1986 *Theory of Group Representations and Applications* (Singapore: World Scientific)
  - [10] Bargman V 1947 *Ann. Math.* **48** 568
  - [11] Vilenkin N Ja and Klimyk A U 1991 *Representations of Lie Groups and Special Functions* (Dordrecht: Kluwer)
  - [12] Sally P J 1967 *Memoirs Am. Math. Soc.* **69** 1
  - [13] Loll R 1990 *Phys. Rev. D* **41** 3785
  - [14] Piechocki W 2001 Preprint gr-qc/0105005